①

Markov properties on undirected graphs

Global: For any triple (A,B,S) of disjoint subsets of V_G such that S separates A from B in G, we have: $A \perp\!\!\!\perp B \mid S$.

Local: $\forall \alpha \in V_G$: $\alpha \perp\!\!\!\perp V \setminus cl(\alpha) \mid bd(\alpha)$

Pairwise: $\alpha \perp\!\!\!\perp \beta \mid V \setminus \{\alpha,\beta\}$, for any pair (α,β) of non-adjacent vertices.

Reference: Graphical Models: Lauritzen 1996. page: 32. You can find the definition of "closure" & "boundary" in page: 6 of that book.

Markov Properties on DAGs

Global: $A \perp\!\!\!\perp B \mid S$, whenever A and B are separated by S in $(G_{An(A\cup B\cup S)})^m$, the moral graph of the smallest ancestral set containing $A\cup B\cup S$.

Local: $\alpha \perp\!\!\!\perp nd(\alpha) \mid pa(\alpha)$ or $\alpha \perp\!\!\!\perp nd(\alpha)\setminus pa(\alpha) \mid pa(\alpha)$

Pairwise: For any pair (α,β) of non-adjacent vertices with $\beta \in nd(\alpha)$: $\alpha \perp\!\!\!\perp \beta \mid nd(\alpha)\setminus\{\beta\}$

Reference: The same as above reference. For definition of ancestral set and non-descendants see page: 6 of that book.

Ordered (well-numbering local) Markov property:

$$\alpha \perp\!\!\!\perp (pr(\alpha)\setminus pa(\alpha)) \mid pa(\alpha).$$

Reference: Independence properties of Markov fields: Lauritzen, Dawid Larsen, Leimer 1990

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Markov Properties for LWF chain graphs

<u>Global</u>: For any triple (A,B,S) of disjoint subsets of V_G s.t. S separates A from B in $(G_{An(A\cup B\cup S)})^m$, the moral graph of the smallest ancestral set containing $A\cup B\cup S$, we have $A \perp\!\!\!\perp B \mid S$.

<u>Local</u>: $\forall \alpha \in V_G$: $\alpha \perp\!\!\!\perp nd(\alpha) \mid bd(\alpha)$ or $\alpha \perp\!\!\!\perp nd(\alpha) \setminus bd(\alpha) \mid bd(\alpha)$

<u>Pairwise</u>: For any pair (α,β) of non-adjacent vertices with $\beta \in nd(\alpha)$: $\alpha \perp\!\!\!\perp \beta \mid nd(\alpha) \setminus \{\beta\}$

<u>Pairwise block-recursive</u>: Let the vertex set be partitioned in a chain components as $V = V(1) \cup \cdots \cup V(T)$. The set of "concurrent" variables relative to this partitioning is defined as $C(t) = V(1) \cup \cdots \cup V(t)$.
For any pair (α,β) of non-adjacent vertices we have:

$$\alpha \perp\!\!\!\perp \beta \mid C(t^*) \setminus \{\alpha,\beta\},$$

where t^* is the smallest $\underline{t}$ that has $\alpha,\beta \in C(t)$.
Reference: Lauritzen 1996: Graphical Models.

<u>Block-Recursive Markov property</u>:

$$\begin{cases} (C1) \quad \tau \perp\!\!\!\perp \left(nd_D(\tau) \setminus pa_D(\tau)\right) \mid pa_D(\tau) & \forall \tau \in \mathcal{T} \\ (C2a) \quad \sigma \perp\!\!\!\perp \left(\tau \setminus Nb_G(\sigma)\right) \mid \left(pa_D(\tau) \cup nb_G(\sigma)\right), & \forall \tau \in \mathcal{T}, \forall \sigma \subseteq \tau \\ (C3a) \quad \sigma \perp\!\!\!\perp \left(pa_D(\tau) \setminus pa_G(\sigma)\right) \mid \left(pa_G(\sigma) \cup nb_G(\sigma)\right), & \forall \tau \in \mathcal{T}, \forall \sigma \subseteq \tau \end{cases}$$

Reference: Drton (2009) paper.

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Markov Properties for MVR chain graphs

<u>Global</u>: $X \perp\!\!\!\perp Y \mid Z$, if X separated from Y by Z in $(G_{ant(X \cup Y \cup Z)})^a$ or $(G_{an(X \cup Y \cup Z)})^a$

<u>Alternative Global</u>: $X \perp\!\!\!\perp Y \mid Z$, if X separated from Y by Z in $(G_{Antec(X \cup Y \cup Z)})^a$

<u>Ordered Local Markov property</u>: For any ancestral set A such that $x \in A \subseteq pre_{G,<}(x)$:

$$x \perp\!\!\!\perp A \setminus (mb(x,A) \cup \{x\}) \mid mb(x,A).$$

[Reference: Markov properties for Acyclic directed mixed graphs, Thomas Richardson 2003]

<u>Local</u>: $x \perp\!\!\!\perp (nd(x) \setminus bd(x)) \mid pa(x)$

<u>pairwise</u>: For every uncoupled nodes i & j:

(P1): $i \perp\!\!\!\perp j \mid pst(i,j)$

(P2): $i \perp\!\!\!\perp j \mid ant(i,j)$

(P3): $i \perp\!\!\!\perp j \mid pa_G(i,j)$

(P4): $i \perp\!\!\!\perp j \mid pa_G(i)$

[Reference: Sadeghi & Wermuth 2016]

For more details and references see my paper.

* <u>Block-Recursive Markov property</u>: (Drton 2009)

$$\begin{cases} (C1)\ \tau \perp\!\!\!\perp (nd_D(\tau) \setminus pa_D(\tau)) \mid pa_D(\tau) & \forall \tau \in \mathcal{T} \\ (C2b)\ \sigma \perp\!\!\!\perp (\tau \setminus Nb_G(\sigma)) \mid pa_D(\tau) & \forall \tau \in \mathcal{T}\ \&\ \forall \sigma \subseteq \tau \\ (C3b)\ \sigma \perp\!\!\!\perp (pa_D(\tau) \setminus pa_G(\sigma)) \mid pa_G(\sigma) & \forall \tau \in \mathcal{T}\ \&\ \forall \sigma \subseteq \tau. \end{cases}$$

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Markov Properties for AMP chain graphs

Global: For any triple (A,B,S) of disjoint subsets of V_G s.t. S separates A from B in $G[A \dot\cup B \dot\cup S]^a$, we have $A \perp\!\!\!\perp B \mid S$.

[Reference: Alternative Markov properties for chain Graphs
Andersson, Madigan, Perlman 2001]

Local:

$$\begin{cases} (L1)\ \forall v \in V:\ v \perp\!\!\!\perp \left(\tau(v) \setminus cl_{G_{\tau(v)}}(v)\right) \mid nd_D(\tau(v)) \dot\cup nb_{G_{\tau(v)}} \\ (L2)\ \forall v \in V:\ v \perp\!\!\!\perp \left(nd_D(\tau(v)) \setminus pa_G(v)\right) \mid pa_G(v) \end{cases}$$

[same references as abovementioned].

Pairwise: (Two Def.)

$$\begin{cases} (P_1):\ \forall v \in V,\ \forall w \in \tau(v) \setminus cl_{G(\tau(v))}(v):\ v \perp\!\!\!\perp w \mid nd_D(\tau(v)) \dot\cup (\tau(v) \setminus \{v,w\}) \\ (P_2):\ \forall v \in V,\ \forall w \in nd_D(\tau(v)) \setminus pa_G(v):\ v \perp\!\!\!\perp w \mid nd_D(\tau(v)) \setminus \{w\}. \end{cases}$$

$$\begin{cases} (P_1'):\ \quad '' \quad :\ v \perp\!\!\!\perp w \mid nd_D(\tau(v)) \dot\cup nb_{G_{\tau(v)}}(v) \\ (P_2'):\ \quad '' \quad :\ v \perp\!\!\!\perp w \mid pa_G(v) \end{cases}$$

[Reference: same]

Block-recursive: (C1): $\tau \perp\!\!\!\perp \left(nd_D(\tau) \setminus pa_D(\tau)\right) \mid pa_D(\tau),\ \forall \tau \in \mathcal{T}$.

(C2a): $\sigma \perp\!\!\!\perp \left(\tau \setminus Nb_G(\sigma)\right) \mid \left(pa_D(\tau) \cup nb_G(\sigma)\right)\quad \forall \tau \in \mathcal{T}\ \&\ \forall \sigma \subseteq \tau$.

(C3b): $\sigma \perp\!\!\!\perp \left(pa_D(\tau) \setminus pa_G(\sigma)\right) \mid pa_G(\sigma)\quad \forall \tau \in \mathcal{T},\ \forall \sigma \subseteq \tau$.

(Reference: Drton, 2009)

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Markov Properties for bi-directed (Covariance) graphs

Global: $B \perp\!\!\!\perp C \mid A$, when B and C are separated by $D = V \setminus \{A, B, C\}$ in G.

Local: $\alpha \perp\!\!\!\perp V \setminus \{\alpha, bd(\alpha)\}$

pairwise: $\alpha \perp\!\!\!\perp \beta$ for all α, β not adjacent in G.

[References: On Dualization of graphical gaussian models. Göran Kauermann 1996]